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1994 J. Phys. A: Math. Gen. 27 3245

(http://iopscience.iop.org/0305-4470/27/9/034)

J. Phys. A: Math. Gen. 27 (1994) 3245–3256. Printed in the UK

Focusing and defocusing of electromagnetic waves in a ferromagnet

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Received 11 October 1993, in final form 24 January 1994

Abstract. We show that the modulation of an electromagnetic wave in a ferromagnet in the presence of an external magnetic field is governed by the nonlinear Schrödinger equation. We characterize the existence of an oscillatory instability of the Benjamin–Feir type for electromagnetic propagation in a saturated ferrite. Depending on the physical parameters of the system, we establish the regions in which focusing or defocusing of the initial carrier envelope occurs. We show that one of the two plane waves, circularly polarized, allowed to propagate into a ferrite, is bunching into solitons and the other is modulated by a dark soliton.

1. Introduction

The study of propagation of electromagnetic waves in ferromagnets is interesting not only from a theoretical point of view but also from a practical point of view, particularly in connection with the behaviour of ferrite devices, such as ferrite-loaded waveguides, at microwave frequencies [1].

The propagation of electromagnetic waves in a ferromagnet obeys nonlinear equations with dispersion and dissipation. The linear theory has been investigated extensively in reference [2] and this approach provided a good explanation for phenomena such as cut-offs and resonances.

Recently Nakata [3] began a rigorous study of the nonlinear case investigating nonlinear propagation of long-wavelength electromagnetic waves in a saturated ferromagnet, taking into account nonlinearity and dispersion. Using a multi-scale expansion method, the Maxwell equations in the ferromagnet were reduced to the modified Korteweg–de Vries equation.

In a previous paper [4] we studied the effects of dissipation and nonlinearity on the propagation of a small electromagnetic perturbation in a saturated ferrite in the presence of an external constant magnetic field. We showed that such dynamics obeys the nonlinear Burgers' equation in (1+1) and (2+1) dimensions.

In this letter, instead of looking for propagation of long-wavelength waves, we investigate a modulational phenomenon. We study how an electromagnetic plane wave is modulated by nonlinear and dispersive effects in a saturated ferromagnet. We confine the study to the case of slow modulation (the change of the wave envelope is slow in both space and time in comparison with the carrier wave), which allows us to use the stretched co-ordinates method and to consider the system without dissipation and in (1+1) dimensions.

We find that the modulation of such waves in the lowest order of perturbation is governed by the nonlinear Schrödinger equation (NLS). This allows us to characterize, in a rigorous fashion, a modulational instability of the Benjamin–Feir type [5, 6]. This instability is governed by the values of some physical parameters in the space of which we determine whether the carrier wave is stable or not.

The main new result obtained is: a plane wave of positive helicity ('left circularly polarized wave' in the optical convention) propagating parallel to the applied field, is destroyed (forming solitons) and a plane wave of negative helicity ('right circularly polarized wave') propagates without bunching, its amplitude being only slowly modulated.

2. Mathematical formulation of the phenomenological model

The general form of Maxwell's equations in MKS units reads

$$\nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1}$$

$$\nabla \wedge \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{2}$$

in which $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$ have their standard meaning. The constitutive equations in the ferromagnet for $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{H}$, $\boldsymbol{B}$ are given by

$$\boldsymbol{D} = \hat{\epsilon}\ \boldsymbol{E} \tag{3}$$

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \tag{4}$$

where we shall assume that $\hat{\epsilon}$ is the scalar permittivity of a ferromagnet, μ_0 is the magnetic permeability in vacuum, and $\boldsymbol{M}$ is the magnetization density in a ferromagnet. We consider a ferromagnet with saturated magnetization density. In the presence of an external magnetic field the magnetization density is governed by the torque equation which, when damping is neglected, becomes [1,2]

$$\frac{\partial \boldsymbol{M}}{\partial t} = -\mu_0 \delta \boldsymbol{M} \wedge \boldsymbol{H} \tag{5}$$

where δ is the gyromagnetic ratio. This equation shows that $\boldsymbol{M}$ is not parallel to $\boldsymbol{H}$ and that its functional relationship to $\boldsymbol{H}$ is not linear. In (5) we do not consider either the term coming from the magnet anisotropy or the one that represents the inhomogeneous exchange interaction. The former is neglected because we consider an isotropic ferromagnet and the latter because the space scale associated with electromagnetic waves in ferrites substantially exceeds the space scale associated with the inhomogeneous exchange interaction (typically that of spin waves).

Taking the curl of equation (2) and using (1), (3) and (4), we have

$$-\nabla(\nabla \cdot \boldsymbol{H}) + \nabla^2 \boldsymbol{H} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\boldsymbol{H} + \boldsymbol{M}) \tag{6}$$

where $c = 1/\sqrt{\hat{\epsilon}\mu_0}$ is the speed of light based on the dielectric constant of the ferromagnet. If the magnetization were zero, $\nabla \cdot \boldsymbol{H} = 0$ and (6) would be the linear wave equation, satisfied by isotropic, dispersionless transverse waves propagating at speed c. This is not the case, so equations (5) and (6) are a system of nonlinear partial differential equations for $\boldsymbol{M}$ and $\boldsymbol{H}$ which admits sinusoidal waves solutions proportional to $A \exp \mathrm{i}[\boldsymbol{kr} - \omega(k)t]$ only if the amplitude A is sufficiently small. We are going to consider harmonic solutions of (5) and (6) in one space coordinate x and time t which, although of small amplitude, are nevertheless large enough so that the effect of nonlinearity cannot be neglected. Nonlinear terms give rise to a modulation of the amplitude as well as waves of higher harmonics. Our aim is to investigate how the amplitude is modulated by nonlinear effects, on the condition that this modulation is slow over the period of the oscillations of the sinusoidal part.

3. Perturbation scheme and the nonlinear Schrödinger equation

Since the amplitude of the wave solution considered is small, we use the smallness of it, names 'ϵ', for characterized all the quantities which we will consider small or slow in the system. We are restricted to situations where the change of the wave envelope is slow in both space (ξ) and time (τ) in comparison with x and t of the carrier wave. Hence, we define a slow time scale $\xi(x,t)$ and a 'coarse-grained' space scale $\tau(x,t)$ satisying

$$\frac{\partial \xi(x,t)}{\partial x} = \epsilon^p \ll 1 \qquad \frac{\partial \tau(x,t)}{\partial t} = \epsilon^q \ll 1.$$

The equation for ξ reads

$$\xi(x,t) = \epsilon^p (x + a(t)).$$

We choose the (free) function of integration $a(t)$ so that the definition of ξ implies looking at the system in a reference frame that move with the group velocity V_g of the carrier wave,

$$\xi = \epsilon^p (x - V_g t).$$

This physical choice allows us to eliminate the linear effects (basic approximation) which move at velocity V_g. We will see later (see equation (19)) that this hypothesis is one of the solvability condition of the system. For τ we have

$$\tau = \epsilon^q t$$

since the arbitrary function of integration in x was fixed equal to zero in order to preserve the Galilean nature of the ξ, τ transformations.

The values of p and q are not arbitrary. Large values of p, q would yield equations which contain divergences in the $\epsilon \to 0$ limit, while too small values for p, q would yield results of no interest in the $\epsilon \to 0$ limit. The choice $p = q/2 = 1$ is the most appropriate one, given a first nonlinear correction at the linear solution.

Let us thus seek a solution of equations (5) and (6) in the form of a Fourier expansion in harmonics of the fundamental $E = exp\{i(kx - \omega t)\}$ as

$$\boldsymbol{M} = \sum_{n=-\infty}^{+\infty} \boldsymbol{M}^n E^n \tag{7a}$$

$$\boldsymbol{H} = \sum_{n=-\infty}^{+\infty} \boldsymbol{H}^n E^n \tag{7b}$$

where the Fourier components are developped in a Taylor series in powers of the small parameter ϵ measuring the normalized amplitude of the applied RF field:

$$\boldsymbol{M}^n = \sum_{j=0}^{\infty} \epsilon^j \boldsymbol{M}_j^n(\xi,\tau) \tag{8a}$$

$$\boldsymbol{H}^n = \sum_{j=0}^{\infty} \epsilon^j \boldsymbol{H}_j^n(\xi,\tau) \tag{8b}$$

Here we have the real-valuedness conditions $\boldsymbol{M}^{-n} = (\boldsymbol{M}^n)^*$ and $\boldsymbol{H}^{-n} = (\boldsymbol{H}^n)^*$, where the asterisk denotes complex conjugation; τ, ξ are slow variables introduced via the stretching

$$\xi = \epsilon(x - Vt) \tag{9a}$$

$$\tau = \epsilon^2 t \tag{9b}$$

where the velocity V will be determined later as a solvability condition of equations (5), (6). Substituting (7*a*), (7*b*) into (5), (6) for $\boldsymbol{M} = (M_x, M_y, M_z)$, $\boldsymbol{H} = (H_x, H_y, H_z)$, rescaling $\boldsymbol{M}, \boldsymbol{H}, t$ into $\frac{\delta\mu_0 \boldsymbol{M}}{c}$, $\frac{\delta\mu_0 \boldsymbol{H}}{c}$, ct and collecting powers of E, we obtain

$$\left(\frac{\partial}{\partial t} - \mathrm{i}n\omega\right)\boldsymbol{M}^n = -\sum_{p+q=n} \boldsymbol{M}^p \wedge \boldsymbol{H}^q \tag{10a}$$

$$\left(\frac{\partial^2}{\partial t^2} - 2\mathrm{i}n\omega\frac{\partial}{\partial t} - n^2\omega^2\right)(H_s^n + M_s^n) = \left(\frac{\partial^2}{\partial x^2} + 2\mathrm{i}nk\frac{\partial}{\partial x} - n^2k^2\right)H_s^n(1-\delta_{s,x}) \tag{10b}$$

with $s = x, y, z$ and $\delta_{s,i} = 1$ for $s = i$ and $\delta_{s,i} = 0$ if $s \neq i$. Introducing now the expansions (8) and the slow variables (9) into (10), we can proceed to collect and solve different orders of ϵ^j and harmonics n (order (j, n)) with the conditions $M_j^n, H_j^n \to 0$ as $\xi \to -\infty$ for $(j, |n|) \neq (0, 0), (1, 1)$. We asssume that $\boldsymbol{M}_0^0 = \boldsymbol{m}$ and $\boldsymbol{H}_0^0 = \boldsymbol{h}$ are constants. The field $\boldsymbol{H}_0^0$ represents the external constant magnetic field in which the ferrite is immersed and $\boldsymbol{M}_0^0 = \boldsymbol{m}$ the magnetization of saturation. The state $\boldsymbol{H}_0^0$, $\boldsymbol{M}_0^0$ represents the initial static state and the developments (8*a*) and (8*b*) represent a perturbation of this state. The vectors $\boldsymbol{H}_1^1$, $\boldsymbol{M}_1^1$ tending to constants for $\xi \to -\infty$ (see expressions 15(*a,b,c*)). For an appropriate choice of the Cartesian coordinate system we can write $\boldsymbol{m}$ as $\boldsymbol{m} = (m_x, m_t, 0)$.

In the leading order $(0, n)$ we have

$$\mathrm{i}n\omega\boldsymbol{M}_0^n = \sum_{p+q=n} \boldsymbol{M}_0^p \wedge \boldsymbol{H}_0^q \tag{11a}$$

$$n^2\omega^2(H_{0,s}^n + M_{0,s}^n) = n^2k^2H_{0,s}^n(1-\delta_{s,x}). \tag{11b}$$

This system has a particular solution

$$\boldsymbol{M}_0^n = \boldsymbol{m}\,\delta_{0,n} \qquad \boldsymbol{H}_0^n = \alpha\boldsymbol{m}\,\delta_{0,n} \tag{12}$$

where α is given by $\boldsymbol{h} = \alpha\boldsymbol{m}$.

In the order $(1, n)$ we obtain (using (12))

$$\mathrm{i}n\omega\boldsymbol{M}_1^n = \boldsymbol{m} \wedge \left(\boldsymbol{H}_1^n - \alpha\boldsymbol{M}_1^n\right) \tag{13a}$$

$$n^2\omega^2\left(H_{1,s}^n + M_{1,s}^n\right) = n^2k^2H_{1,s}^n(1-\delta_{s,x}). \tag{13b}$$

Equations (13*b*) give the components $M_{1,s}^n$ as functions of $H_{1,s}^n$. Using this in (13*a*), we find a linear homogeneous system for $H_{1,x}^n$, $H_{1,y}^n$, $H_{1,z}^n$. If $n = 0$, we can choose the solution of (13) as $\boldsymbol{M}_1^0 = \boldsymbol{H}_1^0 = 0$. For the order (1,1) the determinant of the system is zero if ω verifies the dispersion relation

$$(\omega^2 - k^2)(\omega^2 + \alpha\omega^2 - \alpha k^2)(1+\alpha)m_t^2 + (\omega^2 + \alpha\omega^2 - \alpha k^2)^2 m_x^2 = (\omega^2 - k^2)^2\omega^2. \tag{14}$$

We must make the hypothesis that the monochromatic RF applied satisfies one of the three solutions $\omega(k)$ of (14).

Under this condition we obtain the following non-trivial solution of (13):

$$M_{1,x}^1 = -H_{1,x}^1 = -\mathrm{i}\gamma\mu m_t g(\xi, \tau) \tag{15a}$$

$$M_{1,y}^1 = -\gamma H_{1,y}^1 = \mathrm{i}\gamma\mu m_x g(\xi, \tau) \tag{15b}$$

$$M_{1,z}^1 = -\gamma H_{1,z}^1 = -\gamma^2\omega g(\xi, \tau) \tag{15c}$$

where $\gamma = 1 - \frac{k^2}{\omega^2}$, $\mu = 1 + \alpha\gamma$ and $g(\xi, \tau)$ is an arbitrary function, such that $|g|^2 \to \lambda \neq 0$ for $\xi \to -\infty$. In this order we can calculate the group velocity of the primary progressive wave. It reads

$$V_g = \frac{\partial \omega}{\partial k} = \frac{u(b+1)}{\gamma \mu u^2 + b + 1} \tag{16}$$

where $b = \mu^2 m_x^2/\gamma^2\omega^2$ and $u = \omega/k$ is the phase velocity. For $|n| > 1$, the determinant of the system is nonzero, so that $M_1^n = \vec{H}_1^n = \mathrm{o}$, for $|n| > 1$. This completes the solution at order $(1, n)$. In the next order, we have the system

$$\begin{aligned} \mathrm{i}n\omega M_2^n - m \wedge \left(H_2^n - \alpha M_2^n\right) &= \left(M_1^1 \wedge H_1^1\right)\delta_{n,2} + \left(M_1^{1*} \wedge H_1^1 + M_1^1 \wedge H_1^{1*}\right)\delta_{n,0} \\ &\quad + \left(M_1^{1*} \times H_1^{1*}\right)\delta_{n,-2} - V\frac{\partial}{\partial\xi}M_1^n \end{aligned} \tag{17a}$$

$$\begin{aligned} -\omega^2 n^2 &\left(H_{2,s}^n + M_{2,s}^n\right) + n^2k^2 H_{2,s}^n(1 - \delta_{1,x}) \\ &= -2\mathrm{i}n\omega V\frac{\partial}{\partial\xi}\left(H_{1,s}^n + M_{1,s}^n\right) + \left[2\mathrm{i}nk\frac{\partial}{\partial\xi}H_{1,s}^n\right](1 - \delta_{1,x}). \end{aligned} \tag{17b}$$

For $|n| > 2$, we obtain an homogeneous system with non-zero determinant; consequently, only the trivial solution exists, so that $M_2^n = H_2^n = 0$ for $|n| > 2$. For the order (2,2) we have an inhomogeneous system in H_2^2 components whose complete solution is given by

$$M_{2,x}^2 = -H_{2,x}^2 = -\frac{\gamma\mu^2 m_x(1-b)}{2u^2(1+\alpha)}g^2 \tag{18a}$$

$$M_{2,y}^2 = -\gamma H_{2,y}^2 = \frac{\gamma^2\mu m_t}{2u^2}(1+b)g^2 \tag{18b}$$

$$M_{2,z}^2 = -\gamma H_{2,z}^2 = \frac{\mathrm{i}\gamma\mu^2 m_x m_t}{u^2\omega}g^2. \tag{18c}$$

Furthermore, in the case (2,1) we have an inhomogeneous linear system for the H_2^1 components, but the determinant of the associated homogeneous systems is in this case zero owing to the dispersion relation (14). Therefore, the system will have a solution if the determinant of augmented matrix is also zero. This condition is satisfied if

$$V = V_g \tag{19}$$

which determines V. Under this solvability condition we get

$$H_{2,x}^1 = -\gamma\mu m_t f + \Omega m_t(b + 1 + 2\alpha\gamma)\frac{\partial g}{\partial\xi} \tag{20a}$$

$$H_{2,y}^1 = \mu m_x f + \frac{\Omega m_x}{\gamma}(1-b)\frac{\partial g}{\partial\xi} \tag{20b}$$

$$H_{2,z}^1 = \mathrm{i}\gamma\omega f \tag{20c}$$

$$M_{2,x}^1 = \gamma\mu m_t f - \Omega m_t(b + 1 + 2\alpha\gamma)\frac{\partial g}{\partial\xi} \tag{20d}$$

$$M_{2,y}^1 = -\gamma\mu m_x f + \Omega m_x(b + 1 + 2\alpha\gamma)\frac{\partial g}{\partial\xi} \tag{20e}$$

$$M_{2,z}^1 = -\mathrm{i}\gamma^2\omega f + 2\mathrm{i}\gamma\omega\Omega\frac{\partial g}{\partial\xi} \tag{20f}$$

where $\Omega = \frac{k^2}{\omega^3}\left(V - \frac{\omega}{k}\right)$ and $f(\xi, \tau)$ is an arbitrary function. At the order (2,0), equations (17a), (17b) do not contain all the necessary information to determine completely $\boldsymbol{M}_2^0, \boldsymbol{H}_2^0$. We must go to the order (4,0) of equation (10b) to determine $\boldsymbol{M}_2^0$ as a function of $\boldsymbol{H}_2^0$. At this order, using the results of the orders $(0, n)$ and $(1, n)$, we obtain the relations

$$M_{2,s}^0 = -H_{2,s}^0(\delta_{s,x} + \beta(1 - \delta_{s,x})), \tag{21}$$

where $\beta = 1 - V^{-2}$. Making use of these equations in (17a, b), we find $\boldsymbol{M}_2^0$, $\boldsymbol{H}_2^0$. They read

$$M_{2,x}^0 = -H_{2,x}^0 = -m_x(1 + \alpha\beta)\,\varphi \tag{22a}$$

$$M_{2,y}^0 = -\beta H_{2,y}^0 = -\beta m_t(1 + \alpha)\,\varphi + \frac{2\gamma\beta(1 - \gamma)}{1 + \alpha\beta}\mu^2 m_t |g|^2 \tag{22b}$$

$$M_{2,z}^0 = H_{2,z}^0 = 0 \tag{22c}$$

where φ is a function that will be determined below.

The next order $(3, n)$ is the laborious one which allows us to find the function $\varphi(\xi, \tau)$ and the nonlinear evolution of $g(\xi, \tau)$. We have the following set of equations:

$$\begin{aligned} n^2\omega^2\left(H_{3,s}^n + M_{3,s}^n\right) - n^2k^2 H_{3,s}^n(1 - \delta_{s,x}) &= 2in\omega V\frac{\partial}{\partial\xi}\left(H_{2,s}^n + M_{2,s}^n\right) \\ &\quad -\left(V^2\frac{\partial^2}{\partial\xi^2} + 2in\omega\frac{\partial}{\partial\tau}\right)(H_{1,s}^n + M_{1,s}^n) \\ &\quad -\left(2ink\frac{\partial}{\partial\xi}H_{2,s}^n + \frac{\partial^2}{\partial\xi^2}H_{1,s}^n\right)(1 - \delta_{s,x}) \end{aligned} \tag{23a}$$

$$\begin{aligned} &in\omega \boldsymbol{M}_3^n - \boldsymbol{m} \wedge \left(\boldsymbol{H}_3^n - \alpha \boldsymbol{M}_3^n\right) \\ &\qquad = \sum_{p+q=n}\left(\boldsymbol{M}_1^p \wedge \boldsymbol{H}_2^q + \boldsymbol{M}_2^p \wedge \boldsymbol{H}_1^q\right) - V\frac{\partial}{\partial\xi}\boldsymbol{M}_2^n + \frac{\partial}{\partial\tau}\boldsymbol{M}_1^n. \end{aligned} \tag{23b}$$

For the order (3,0) the equation (23b) has a solution only if

$$\boldsymbol{m}\left\{2\,\mathrm{Re}\left[\boldsymbol{M}_1^1 \wedge \boldsymbol{H}_2^{1*} + \boldsymbol{M}_2^{1*} \wedge \boldsymbol{H}_2^1\right] - V\frac{\partial}{\partial\xi}\boldsymbol{M}_2^0\right\} = 0. \tag{24}$$

This equation determines $\varphi(\xi, \tau)$ in terms of $g(\xi, \tau)$:

$$\varphi(\xi, \tau) = \frac{1}{d}\left[\frac{2\beta\gamma(1 - \gamma)\mu^2}{1 + \alpha\beta}m_t^2 - \frac{\gamma\omega\Omega}{V}\Lambda\mu\right]|g|^2 + \frac{\gamma\omega\Omega\Lambda\mu\lambda}{Vd} \tag{25}$$

with

$$d = m_x^2(1 + \alpha\beta) + (1 + \alpha)\beta m_t^2 \tag{26}$$

$$\Lambda = \frac{\Gamma}{\gamma\mu^2(1 + \alpha)}\left[2b\gamma(1 + \alpha) + 2\mu - (b^2 + 1)(1 - \gamma)\right] \tag{27}$$

$$\Gamma = \gamma^2\omega^2. \tag{28}$$

For the order (3,1) we obtain an inhomogeneous linear system of equations for the components of $\boldsymbol{H}_3^1$. The compatibility conditions for this system give a nonlinear evolution equation for $g(\xi, \tau)$ in which the term f coming from $\boldsymbol{H}_2^1, \boldsymbol{M}_2^1$ is eliminated using (14). It reads

$$iA\frac{\partial g}{\partial\tau} + B\frac{\partial^2 g}{\partial\xi^2} + Cg|g|^2 + D\lambda g = 0 \tag{29}$$

with the condition $|g|^2 \to \lambda$ for $\xi \to -\infty$; the real constants A, B, C, D are given by

$$A = -\frac{2\Gamma\omega}{\mu u^2}(b + 1 + \gamma\mu u^2) \tag{30}$$

$$B = \frac{\Gamma\gamma u^2}{(b + 1 + \gamma\mu u^2)^2}\mathcal{P} \tag{31}$$

$$C = \frac{\Gamma^2\mu}{2(1+\alpha)}\frac{\mathcal{L}}{\mathcal{Q}} \tag{32}$$

$$D = \frac{\Gamma^3[2\mu - (b+1)(1-\gamma)]}{(1+\alpha)(b+1)^2 d}\mathcal{H} \tag{33}$$

with $\mathcal{P}, \mathcal{Q}, \mathcal{H}, \mathcal{L}$ given by

$$\mathcal{P} = 3b^2 - b - (b+1)(3\alpha\gamma + \mu u^2) \tag{34}$$

$$\mathcal{Q} = (b+1)^2 - (2(b+1) + \gamma\mu u^2)(\mu - b) \tag{35}$$

$$\mathcal{H} = (1-\gamma)(1-3b)(b+1)^2 + \gamma[2(b+1) + \gamma\mu u^2][(\alpha+1)(b+1) + \mu(1-3b)] \tag{36}$$

$$\mathcal{L} = (b+1)^2\mathcal{B}_1 - (2(b+1) + \gamma\mu u^2)\mathcal{B}_2 \tag{37}$$

with

$$\mathcal{B}_1 = (1-\gamma)\Big\{(1-\gamma)[15b^2 - 6b - 1] + 4\mu(1-3b)\Big\} \tag{38}$$

$$\begin{aligned}\mathcal{B}_2 = 2\gamma(b+1)&\Big\{(1-\gamma)[(1-3b)\mu + (1+\alpha)(b+1)] - 2\mu(1+\alpha)\Big\}\\ &+(1-\gamma)(1-b)\Big\{-3b^2(1-\gamma) + b[(1-\gamma)(3\mu - 5)\\ &+4(3\gamma - 1)\mu] + \mu(1-\gamma)\Big\} - 4\gamma\mu^2(1-3b).\end{aligned} \tag{39}$$

We can now make the transformation

$$g(\xi, \tau) = \varphi(\xi, \tau)\exp\left(\mathrm{i}\frac{D}{A}\lambda\tau\right) \tag{40}$$

$$T = \frac{B}{A}\tau \qquad X = \xi \qquad E = \frac{C}{B} \tag{41}$$

and we obtain

$$\mathrm{i}\varphi_T + \varphi_{XX} + E\varphi|\varphi|^2 = 0. \tag{42}$$

Equation (29) (or (42)) is the nonlinear Schrödinger (NLS) equation [7] which appears in many branches of physics when nonlinear modulation of waves is studied. This equation has been extensively studied by several methods and we know that it belongs to the class of soliton equations. The nature of solutions of NLS as well as its physical meaning depend drastically on the sign of BC (or E) [8]. For $BC > 0$ we know that the incident carrier wave is destroyed by nonlinearity and dispersion and that it bunches into solitons (focusing case). For $BC < 0$, the incident carrier wave evolves without bunching in a self-similar forms. These two cases characterize regions in the space of the physical parameters (scalar permittivity, magnetic permeability, gyromagnetic ratio, values of the DC applied field, frequency of the RF field etc) where stability or instability of the incident carrier wave occurs. This instability in electromagnetic propagation in a saturated ferrite is reminiscent of the Benjamin–Feir instability phenomenon of the Stokes wave train in water theory [6]. Finaly we note that in contrast to Nakata's result [3], equation (29), or (42), is valid for all angles φ between the direction of propagation of the carrier wave and the external magnetic field. The ordering with respect to ϵ in Nakata's work breaks down because he considered the propagation of long waves (see particularly formula (3.3) in [3]) and this is not the case here.

4. One-envelope soliton solution and one dark-soliton solution: focusing and defocusing of electromagnetic waves in a ferrite

The BC product in (29) can be written as

$$BC = \theta\gamma\mu\frac{\mathcal{L}\mathcal{P}}{\mathcal{Q}} \tag{43}$$

with θ given by

$$\theta = \frac{\Gamma^3 u^2}{2(1+\alpha)(b+1+\gamma\mu u^2)^2}. \tag{44}$$

The sign of the BC product is determined by the values of the three positive parameters

$$u = \frac{\omega}{\kappa} \tag{45}$$

$$\alpha = \frac{|H_0^0|}{|M_0^0|} \tag{46}$$

$$b = \frac{\mu^2 m^2 \cos^2\varphi}{\gamma^2\omega^2}. \tag{47}$$

The first equality is the phase velocity, the second one measures the relative intensity between the external (constant) magnetic field (H_0^0) and the magnetization of saturation (M_0^0), and the last one is related to the angle between the direction of propagation of the carrier wave and the external magnetic field H_0^0. Note that b depends also on α and u.

There are several regions (in the (b, u) plane for given α) where BC is positive, zero or negative. In general we are obliged to make a numerical and approximative study of the expressions for B and C to determine this sign; only in one particular case we can calculate the sign of BC explicitly and without approximation.

In this section we show, omitting all the steps of the associated numerical analysis, the results for the sign of BC in the general case, and we develop, in detail, the exact case.

From the dispersion relation (14), which can be written as

$$\gamma\mu(1+\alpha)m_t^2 = (1-b)\Gamma \tag{48}$$

$$\Gamma = \gamma^2\omega^2 \tag{49}$$

we see that the sign and the values of b are determined from those of u, through the values of γ ($\gamma = 1 - u^{-2}$) and μ ($\mu = 1 + \alpha(1 - u^{-2})$) with $\alpha > 0$ given. We consider (for simplicity) $u \in]0, 1[$. We see that if

$$\gamma\mu > 0 \quad \text{thus} \quad b \in [0, 1] \tag{50a}$$

$$\gamma\mu < 0 \quad \text{thus} \quad b \in]1, \infty[. \tag{50b}$$

Table 1 summarizes the results of the sign of $\gamma\mu$ and the values of b in function of the values of u for $u \in]0, 1]$.

An analytical and numerical study of $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{P}$ (using table 1) allows us to determine the regions where BC is positive, zero or negative. The final results are shown in figure 1. In the regions in-going plane waves are unstable where $BC > 0$, and waves are stable where $BC < 0$.

Table 1. The sign of the product $\gamma\mu$ for $u \in]0, 1]$ and the corresponding values of b.

Values of u	$u \in]0, \sqrt{\frac{\alpha}{1+\alpha}}[$	$u = \sqrt{\frac{\alpha}{1+\alpha}}$	$u \in]\sqrt{\frac{\alpha}{1+\alpha}}, 1[$	$u = 1$
sign of γ	$-$	$-$	$-$	$\gamma = 0$
sign of μ	$-$	$\mu = 0$	$+$	$+$
sign of $\gamma\mu$	$+$	$\gamma\mu = 0$	$-$	$\gamma\mu = 0$
values of b	$b \in [0, 1]$	$b = 0$	$b \in [0, \infty]$	b not defined

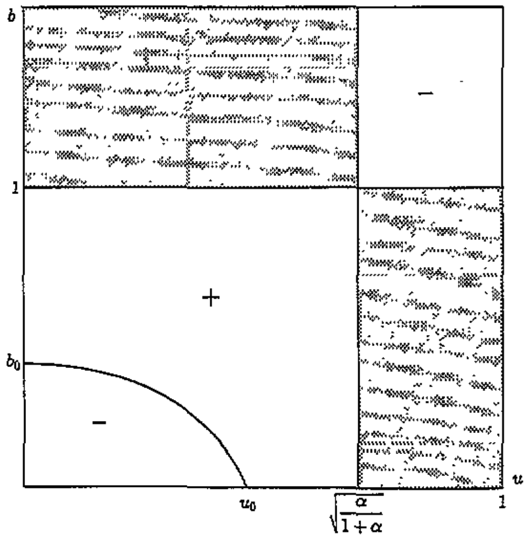

Figure 1. The plane (b, u) for $u \in [0, 1]$, with indication of the sign of the product BC.

The dotted areas are the prohibited regions in the (b, u) plane. The signs of the BC product are indicated inside each permitted region in this plane. The quantity b_0 is determined for a given α by the equation

$$b_0 = \frac{3(1-2\alpha) + 2\sqrt{3(2+2\alpha+3\alpha^2)}}{15} \tag{51}$$

and u_0 is determined numerically, as the root of the equation

$$\mathcal{L}(b = 0, \alpha, u) = 0 \tag{52a}$$

contained between zero and $\sqrt{\frac{\alpha}{1+\alpha}}$. Its asymptotic value for $\alpha \gg 1$ is

$$u_0 \sim \frac{1}{\sqrt{2}}\left(1 - \frac{3}{4\alpha}\right). \tag{52b}$$

The curve joining b_0 to u_0 is given by $\mathcal{L}(b, u, \alpha) = 0$; for $\alpha \gg 1$, its asymptotic equation is given by

$$b = 1 - \frac{2}{3 - 2u^2}. \tag{52c}$$

Let us consider now the only case for which an analytical exact solution can be found. It is represented by the straight line $b = 1$ in the (b, u) diagram and it corresponds to $\varphi = 0$. The direction of propagation of the incident wave is parallel to the external magnetic field $\boldsymbol{H}_0^0$.

Introducing the dimensionless parameter ν ($\nu > 0$) by

$$\nu = \frac{\omega}{m} \tag{53}$$

and solving the dispersion relation (14) for k, we obtain

$$k_{\pm} = \omega \sqrt{\frac{2\alpha(1+\alpha) + (1+\alpha)\sin^2\varphi - 2\nu^2 \pm \sqrt{(1+\alpha)^2\sin^4\varphi + 4\nu^2\cos^2\varphi}}{2\left[\alpha(\alpha+\sin^2\varphi) - \nu^2\right]}}. \tag{54}$$

These two possible values of k represent two elliptically polarized waves propagating in the same direction but with different velocities. In the case studied (longitudinal propagation: $\varphi = 0$, $m^y = 0$, $m^x = m$), we introduce $\epsilon = \pm 1$ and arrive at

$$k_{-\epsilon} = \omega \sqrt{\frac{\nu + \epsilon(1+\alpha)}{\nu + \epsilon\alpha}}. \tag{55}$$

These expressions specify the two possible circularly polarized plane waves traveling in the ferrite. We assume, in order to avoid the two-wave interaction phenomenon, that the in-going plane wave satisfies one of the two solutions (55). Hence, the in-going wave is represented by one of the three branches in figure 2. In the optics terminology, $k_- = P$ represents a left circularly polarized wave and $k_+ = N$ a right circularly polarized wave. Also, such waves are said to have positive helicity or negative helicity, respectively.

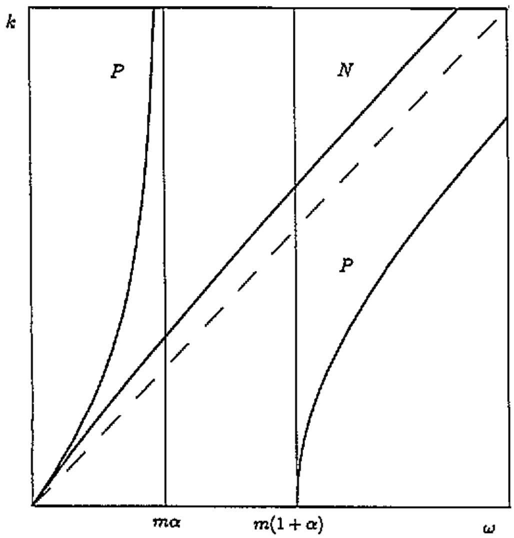

Figure 2. The only two possible circularly polarized plane waves, P or N, allowed to propagate longitudinally in a saturated ferrite.

We also know that

$$u = \sqrt{\frac{\nu + \epsilon\alpha}{\nu + \epsilon(1+\alpha)}} \tag{56}$$

$$\gamma = \frac{-1}{\alpha + \epsilon\nu} \tag{57}$$

$$\mu = \frac{\epsilon\nu}{\alpha + \epsilon\nu}. \tag{58}$$

The coefficients A, B and C have the following simple expression:

$$A = -\frac{2\epsilon m^3 \nu^2}{(\alpha+\epsilon\nu)^3}[2(\alpha+\epsilon\nu)(\alpha+\epsilon\nu+1)-\epsilon\nu] \tag{59a}$$

$$B = -2m^2\nu^2 \frac{[(\alpha+\epsilon\nu)(1+4\alpha)+3\alpha]}{(\alpha+\epsilon\nu)[2(\alpha+\epsilon\nu)(\alpha+\epsilon\nu+1)-\epsilon\nu]^2} \tag{59b}$$

$$C = 4\epsilon m^4 \nu^5 \frac{(\alpha+\epsilon\nu+1)}{(\alpha+\epsilon\nu)^7}. \tag{59c}$$

In this case the sign of BC is determined by the sign of the quantity

$$-\epsilon[\alpha+\epsilon\nu+1][(\alpha+\epsilon\nu)(1+4\alpha)+3\alpha]. \tag{60}$$

Let us set $\epsilon = 1$. This case corresponds to the branch N in figure 2. For $\nu \in [0, \infty[$, we obtain u between $\sqrt{\alpha/(\alpha+1)}$ and 1 and have $BC < 0$, which is easyly seen in figure 1.

If we take $\epsilon = -1$ (the two branches P in figure 2), we have of (60) that $BC < 0$ if ν belongs to $\left[(1+\alpha)4\alpha/(1+4\alpha), 1+\alpha\right]$ and $BC > 0$ if it does not. On the other hand, ν does not belong to $[\alpha, \alpha+1]$ since u must be real. Thus BC is always positive in this case and isd represented by the straight line $b = 1$, with $u \in \left[0, \sqrt{\alpha/(\alpha+1)}\right]$ in figure 1. The values of u between $[1, \infty[$ belong to this case but we do not consider them here.

We can now give some typical solutions of the equations obtained: first in the $BC < 0$ case (the N branch in the longitudinal case), the amplitude of the in-going (carrier) plane wave at $\xi \to -\infty$ is slowly modulated, in the form of a tanh function. The expression of the corresponding dark soliton of (37), calculated using the Hirota method [9] and representing the defocusing case, is given by

$$\varphi(X,T) = \sqrt{\lambda}\exp \mathrm{i}\left[kX-(k^2+2\rho^2)T\right] \times \frac{R^2|B|}{2\lambda|C|}(1+\mathrm{i}r)\left\{\tanh\left[\frac{1}{2}RX+R^2\left(\frac{2k}{R}+r\right)T+\mathrm{i}r\right]\right\} \tag{61}$$

with

$$\rho = \sqrt{\left|\frac{\lambda C}{2B}\right|} \tag{62}$$

$$r = \pm\sqrt{\frac{2\lambda|C|}{R^2|B|}-1} \tag{63}$$

and arbitrary constants R, k and R satisfying

$$|R| < 2\rho. \tag{64}$$

We have thus shown that plane waves having negative helicity in a saturated ferrite are modulationally stable.

On the other hand, in the $BC > 0$ case (the P branch in the longitudinal case), the focusing of the wave envelope occurs and some given initial data bunch into solitons of the expression

$$\varphi(X,T) = \sqrt{\frac{8B}{C}}2\mathrm{i}\eta\frac{\exp\left[-2\mathrm{i}\zeta X-4\mathrm{i}(\zeta^2-\eta^2)T\right]}{ch\ 2\eta(X-X_0+4\zeta T)} \tag{65}$$

where η, ξ, X_0 are arbitrary constants.

We have thus shown that plane waves having positive helicity in a saturated ferrite are unstable.

Using (61) or (65) in (40) and thus in $(15a), (15b), (15c)$, we can obtain the explicit form of solutions for $M^1_{1,s}$ and $H^1_{1,s}$, $s = x, y, z$.

5. Summary, conclusions and perspectives

We have studied the modulation of an electromagnetic wave in a saturated ferromagnet in the presence of an external magnetic field. We have shown that this modulation is governed by the NLS equation. Envelope soliton solutions or dark-soliton solutions of this NLS exist only if its coefficients belong to a determined set of values in a given space of physical parameters. We have established these regions. The very important particular case of longitudinal propagation of plane waves was studied in detail and we have shown that waves with positive helicity are unstable and waves with negative helicity are stable. These facts determine for the first time a Benjamin–Feir instability phenomenon in electromagnetic propagation in a saturated ferrite.

Macroscopic equations (5) and (6), describing the dynamic of electromagnetic-spin waves, are frequently used in the theoretical or experimental approach of waves in ferrites. They have the advantage of maximum fractability and provides a simple phenomenological description of periodic electromagnetic phenomena in a saturated ferrite.

However, they do not provide a complete description of a real saturated ferrite, firstly because we had made some approximations (for exemple, damping due to dissipation was neglected and magnetic anisotropy and inhomogeneous exchange interaction was disregarded) and secondly because this phenomenological description breaks down (just as all theories which involve the use of macroscopic mean-field approximation like $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ in Maxwell equations) when microscopic quantum mechanical effects dominate the macroscopics one. Nevertheless, there are many ferromagnetic materials for which anisotropy forces and damping are negligible. Thus, under conditions in which the mean field approximation is valid, our theoretical conclusions will be experimentally observable. We hope that this work will initiate such experimental studies.

We have left for the future an analysis of the (2+1)-dimensional case and the inclusion of dissipation in the model (Landau damping) which would lead to the nonlinear Davey–Stewartson equation and the nonlinear Ginsburg–Landau equation, respectively.

Acknowledgments

The authors thank Professor J Leon for useful suggestions.

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